Eur. Phys. J. B **41**, 337–343 (2004)
DOI: 10.1140/epjb/e2004-00327-2

THE EUROPEAN
PHYSICAL JOURNAL B

Hopf bifurcation and chaos in a single inertial neuron model with time delay

Chunguang Li[1,a], Guangrong Chen[2], Xiaofeng Liao[1], and Juebang Yu[1]

[1] Institute of Electronic Systems, School of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu, 610054, P.R. China
[2] Department of Electronic Engineering, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong, P.R. China

Received 19 December 2003
Published online 21 October 2004 –

Abstract. A delayed differential equation modelling a single neuron with inertial term subject to time delay is considered in this paper. Hopf bifurcation is studied by using the normal form theory of retarded functional differential equations. When adopting a nonmonotonic activation function, chaotic behavior is observed. Phase plots, waveform plots, and power spectra are presented to confirm the chaoticity.

PACS. 05.45.-a Nonlinear dynamics and nonlinear dynamical systems

1 Introduction

In recent years, dynamical characteristics of neural networks have become a focal subject of intensive research studies. Bifurcations and chaotic phenomena have been investigated in various neural networks. For example, chaotic solutions were obtained in a neural network consisting of 26 neurons in [1]. Numerical solutions of differential equations with electronic circuit models of chaotic neural networks were qualitatively studied in [2]. In [3], a chaotic neural network with four neurons was investigated. Chaotic behavior was found in a cellular neural network with three cells in [4]. In [5], chaotic phenomenon in a three-neuron hysteretic Hopfield-type neural network was discussed. In [6], a high-dimensional chaotic neural network under external sinusoidal force was studied. In [7], bifurcation and chaos as well as their control in a system of strongly connected neural oscillators were discussed. In [8,9], a discrete-time transiently chaotic neural network was studied. The chaotic phenomenon in a neural network learning algorithm was reported in [10]. Moreover, chaotic behaviors of inertial neural networks are studied in [11,12]. On the other hand, there are extensive literatures studied neural network models with delays. For example, bifurcations and chaotic dynamics of neural networks with discrete and distributed delays were studied in [13–21].

In this paper, the dynamical behaviors of a single delayed neuron model with inertial terms are investigated. The work presented in this paper can be considered as an extension of the works for inertial neural network without delays [11,12] to the case with delays, or an extension of the work for single neuron without inertial terms [13] to the case with inertial terms.

The paper is organized as follows. The delayed inertial neuron model is described, and the local stability and the existence of Hopf bifurcation is studied in Section 2. In Section 3, the properties of the bifurcating periodic solutions are analyzed based on the normal form theory developed in [22]. To justify the theoretical analysis, a numerical example is given in Section 4. In Section 5, the observed chaotic behavior of the model with a nonmonotonic activation function is reported. Finally, conclusions are drawn in Section 6.

2 Local stability and the existence of Hopf bifurcation

The single inertial neuron with time delay, similar to that in [13] but with an inertial term, is described by

$$\ddot{x} = -a\dot{x} - bx + cf(x - hx(t-\tau)) \tag{1}$$

where constants $a, b, c > 0, h \geq 0$, and $\tau > 0$ is the time delay. Without loss of generality, assume that the activation function $f(.)$ in the above equation is a nonlinear function and its third-order continuous derivative exists. In [27,28], the authors studied the presence of limit cycles, two-tori and multistability in a damped harmonic oscillator with delayed negative feedback, which is structurally similar to system (1). But the function f in that model is a special delayed feedback function, while in our model, it

[a] e-mail: cgli@uestc.edu.cn

$$L(\varphi) = \begin{bmatrix} \varphi_2(0) \\ -(b - cf'(0))\varphi_1(0) - a\varphi_2(0) - chf'(0)\varphi_1(-\tau) \end{bmatrix}$$
$$F(\varphi) = \begin{bmatrix} 0 \\ \frac{cf''(0)}{2}\varphi_1^2(0) + \frac{cf'''(0)}{6}\varphi_1^3(0) - \frac{chf''(0)}{2}\varphi_1^2(-\tau) - \frac{chf'''(0)}{6}\varphi_1^3(-\tau) + \cdots \end{bmatrix} \quad (5)$$

is a general nonlinear function. Define

$$x_1(t) \equiv x(t) - hx(t-\tau), \qquad t \in [-\tau, +\infty).$$

Then, we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -ax_2(t) - bx_1(t) + cf(x_1(t)) - chf(x_1(t-\tau)). \end{aligned} \quad (2)$$

The phase space is $C := C([-\tau, 0]; R^2)$. Throughout this paper, assume that the following conditions are satisfied:

$$f(0) = 0,\ f'(0) > 0,\ |c(1-h)| < 1/f'(0).$$

It is clear that (2) has an unique equilibrium (0, 0) under the above condition. It is also easy to see that if there is no delay term in (1), i.e. $h = 0$, then the model is asymptotically stable when

$$b - cf'(0) > 0. \quad (3)$$

In the following, we estimate the value of h that preserves the system stability under the above condition.

For convenience, we restate here a result of Bellman and Cooke [24, Theorem 13.9].

Lemma 1 [24]: Let $H(z) \equiv (z^2 + pz + q)e^z + r$, where p is real and positive, q is real and nonnegative, and r is real. Denote by a_k ($k \geq 0$) the sole root of the equation

$$\cot a = \left(a^2 - q\right)/ap$$

which lies on the interval $(k\pi, k\pi + \pi)$. Define the natural number n as follows:

1. if $r \geq 0$ and $p^2 \geq 2q$, $n = 1$;
2. if $r \geq 0$ and $p^2 < 2q$, n is the odd integer k for which a_k lies closest to $\sqrt{q - p^2/2}$;
3. if $r < 0$ and $p^2 \geq 2q$, $n = 2$;
4. if $r < 0$ and $p^2 < 2q$, n is the even integer k for which a_k lies closest to $\sqrt{q - p^2/2}$.

Then, a necessary and sufficient condition under which all the roots of $H(z) = 0$ lie to the left of the imaginary axis is that

1. $r \geq 0$ and $(r \sin a_n)/pa_n < 1$, or
2. $-q < r < 0$ and $(r \sin a_n)/pa_n < 1$.

Separating the linear and the nonlinear terms, (2) becomes

$$\dot{x} = L(x_t) + F(x_t) \quad (4)$$

where $x_t \in C, x_t(\theta) = x(t+\theta), -\tau \leq \theta \leq 0$, and $L : C \to R^2$, $F : C \to R^2$ are given respectively by

See equation (5) above

with $\varphi = (\varphi_1, \varphi_2) \in C$. Here and throughout this paper, we refer to [25] for notation and classical results on functional differential equations (FDEs), including such as equation (4).

The characteristic equation for the linearization of equation (4) at (0, 0) is

$$\lambda^2 + a\lambda + (b - cf'(0)) + chf'(0)e^{-\lambda\tau} = 0. \quad (6)$$

Let $s = \lambda\tau$. Then we have

$$\left[s^2 + a\tau s + (b - cf'(0))\tau^2\right] e^s + chf'(0)\tau^2 = 0. \quad (7)$$

The fixed point is locally stable if all roots of the above equation have negative real parts [25]. For each τ, we are interested in the maximum value of h such that the system is locally stable.

Theorem 1: Denote by w_k ($k \geq 0$) the sole root of the equation

$$\cot w = [w^2 - (b - cf'(0))\tau^2]/a\tau w$$

which lies on the interval $(k\pi, k\pi + \pi)$. Define the nature number n as follows:

1. if $a^2 \geq 2(b - cf'(0))$, $n = 1$;
2. if $a^2 < 2(b - cf'(0))$, n is the odd k for which w_k lies closest to $\sqrt{(b - cf'(0)) - a^2/2}\,\tau$.

Then, under condition (3), a necessary and sufficient condition that the solution of (2) is asymptotically stable is that

$$h < \frac{aw_n}{cf'(0)\tau \sin w_n}.$$

Proof: Since $a > 0, \tau > 0, f'(0) > 0, b - cf'(0) > 0, h \geq 0$, a direct application of Lemma 1 to (7) with $p = a\tau, q = (b - cf'(0))\tau^2$ and $r = chf'(0)\tau^2$ proves the claim.

In the following, we study the existence of Hopf bifurcation in equation (2) by choosing h as the bifurcation parameter. First, we would like to know when equation (6) has purely imaginary roots $\lambda = \pm i\omega_0\,(\omega_0 > 0)$ at $h = h_0$. If $\lambda = \pm i\omega_0$, $\omega_0 > 0$, we have

$$chf'(0)\cos\omega_0\tau = \omega_0^2 - (b - cf'(0))$$
$$chf'(0)\sin\omega_0\tau = a\omega_0.$$

The above equations imply that

$$\cot\omega_0\tau = \frac{\omega_0^2 - (b - cf'(0))}{a\omega_0} \equiv g(\omega_0).$$

and, consequently,

$$g'(\omega) = \frac{\omega^2 + (b - cf'(0))}{a\omega^2} > 0.$$

So, $g(\omega)$ is strictly monotonically increasing on $(0, +\infty)$, with $\lim_{\omega\to 0} g(\omega) = -\infty$ and $\lim_{\omega\to+\infty} g(\omega) = +\infty$. Clearly, $g(\omega)$ intersects $\cot\omega\tau$ only at a point. Hence, $\lambda = \pm i\omega_0$ are simple roots of equation (6), where ω_0 is the unique root of $\cot\omega\tau = \frac{\omega^2-(b-cf'(0))}{a\omega}$, and $h_0 = \frac{a\omega_0}{cf'(0)\sin\omega_0\tau}$.

From [26], we know that all the other roots of equation (6) have negative real parts. We proceed to calculate $\mathrm{Re}[d\lambda/dh]$ at $h = h_0$. Differentiating equation (6) with respect to h yields

$$\frac{d\lambda}{dh} = \frac{cf'(0)e^{-\lambda\tau}}{ch\tau f'(0)e^{-\lambda\tau} - 2\lambda - a}.$$

So, we have

$$\mathrm{Re}\left[\frac{d\lambda}{dh}\right]_{\substack{\lambda=i\omega_0\\ h=h_0}} = \frac{cf'(0)[cf'(0)h_0\tau - a\cos\omega_0\tau + 2\omega_0\sin\omega_0\tau]}{(ch_0\tau f'(0)\cos\omega_0\tau - a)^2 + (ch_0\tau f'(0)\sin\omega_0\tau + 2\omega_0)^2}.$$

From the above analysis, we have the following result.

Theorem 2: Under condition (3), if $cf'(0)h_0\tau - a\cos\omega_0\tau + 2\omega_0\sin\omega_0\tau \neq 0$, then as h pass through the critical value $h_0 = \frac{a\omega_0}{cf'(0)\sin\omega_0\tau}$, there is a Hopf bifurcation of system (1) at its equilibrium (0, 0), where ω_0 is the sole root of $\cot\omega\tau = \frac{\omega^2-(b-cf'(0))}{a\omega}$.

Remark: Note that if we let $w = \omega_0\tau$, then the constant h_0 in Theorem 2 can be rewritten as $h_0 = \frac{aw}{cf'(0)\tau\sin w}$, which is consistent to that in Theorem 1.

3 Direction and stability of bifurcating periodic solutions

In this section, we study the direction and stability of the bifurcating periodic solutions. The method used here is based on the normal form theory developed by Faria and Magalhães [22]. This method computes normal forms for retarded functional differential equations, without computing beforehand the center manifold of the singularity.

As in [21], in the following we assume $f''(0) = 0, f'''(0) \neq 0$. Define $\Lambda = \{-i\omega_0, i\omega_0\}$ and introduce the new parameter $\beta = h - h_0$. Equation (4) can be rewritten as

$$\dot{x} = L_0(x_t) + F_0(x_t, \beta) \tag{8}$$

where

$$L_0(\varphi) = \begin{bmatrix} \varphi_2(0) \\ -(b - cf'(0))\varphi_1(0) - a\varphi_2(0) - ch_0 f'(0)\varphi_1(-\tau) \end{bmatrix}$$

$$F_0(\varphi) = \begin{bmatrix} 0 \\ -cf'(0)\beta\varphi_1(-\tau) + \frac{cf'''(0)}{6}\varphi_1^3(0) - \frac{cf'''(0)}{6}(h_0+\beta)\varphi_1^3(-\tau) + \cdots \end{bmatrix}.$$

Following the formal adjoint theory of FDEs [25], let the phase space C be decomposed according to Λ as $C = P \bigoplus Q$, where P is the center space for $\dot{x} = L_0(x_t)$, i.e., P is the generalized eigenspace associated with Λ. Consider the bilinear form $(\cdot,\cdot)$ associated with the linear equation $\dot{x} = L_0(x_t)$ [23]. Let Φ and Ψ be bases for P and P^* associated with the eigenvalues $\pm i\omega_0$ of the adjoint equation, respectively, and normalize them so that $(\Phi, \Psi) = I$. In complex coordinates, Φ, Ψ are written as 2×2 matrices of the form

$$\begin{aligned} &\Phi(\theta) = [\phi_1(\theta), \phi_2(\theta)], \quad \phi_1(\theta) = e^{i\omega_0\theta}v, \\ &\phi_2(\theta) = \overline{\phi_1(\theta)}, \quad -\tau \le \theta \le 0, \\ &\Psi(s) = \begin{bmatrix} \psi_1(s) \\ \psi_2(s) \end{bmatrix}, \quad \psi_1(s) = e^{-i\omega_0 s}u^T, \\ &\psi_2(s) = \overline{\psi_1(s)}, \quad 0 \le s \le \tau \end{aligned} \tag{9}$$

where the bar means complex conjugation, u^T is the transpose of u, and u, v are vectors in C^2 such that

$$L_0(\phi_1) = i\omega_0 v, \quad u^T L_0(e^{i\omega_0 s}I) = i\omega_0 u^T, \quad (\psi_1, \phi_1) = 1. \tag{10}$$

Note that $\dot{\Phi} = \Phi B$, where B is the diagonal matrix $B = \mathrm{diag}(i\omega_0, -i\omega_0)$. From (10), we have

$$\begin{aligned} &v_2 = i\omega_0 v_1, \left[-(b - cf'(0)) - ch_0 f'(0)e^{-i\omega_0\tau}\right] v_1 \\ &\quad = (a + i\omega_0)v_2, \\ &u_1 = (a + i\omega_0)u_2, \left[-(b - cf'(0)) - ch_0 f'(0)e^{-i\omega_0\tau}\right] u_2 \\ &\quad = i\omega_0 u_1. \end{aligned} \tag{11}$$

Hence, we can select

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i\omega_0 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ \frac{1}{a+i\omega_0} \end{bmatrix} \tag{12}$$

with

$$u_1 = \frac{a + i\omega_0}{a + 2i\omega_0 - ch_0 f'(0)\tau e^{-i\omega_0\tau}}.$$

Here and in the following, we refer to [22] for results and explanations of several notations involved. Enlarging the phase space C by considering the space BC and using the decomposition $x_t = \Phi z(t) + y_t, z \in C^2, y_t \in Q^1$, we decompose system (8) as

$$\begin{cases} \dot{z} = Bz + \Psi(0)F_0(\Phi z + y, \beta) \\ \dot{y} = A_{Q^1}y + (I - \pi)X_0F_0(\Phi z + y, \beta). \end{cases} \tag{13}$$

Consider the Taylor formula

$$\Psi(0)F_0(\Phi z + y, \beta) = \frac{1}{2}f_2^1(z, y, \beta) + \frac{1}{6}f_3^1(z, y, \beta) + h.o.t.$$

where $f_j^1(z, y, \beta)\,(j = 2, 3)$ are homogeneous polynomials in (z, y, β) of degree j with coefficients in C^2 and $h.o.t.$ stands for higher order terms. The normal form on the 2-dimensional center manifold of the origin and $\beta = 0$ is given by

$$\dot{z} = Bz + \frac{1}{2}g_2^1(z, 0, \beta) + \frac{1}{6}g_3^1(z, 0, \beta) + h.o.t. \tag{14}$$

where g_2^1, g_3^1 are second and third order terms in (z, β), respectively.

Using the notations in [22], we have

$$\frac{1}{2}g_2^1(z, 0, \beta) = \frac{1}{2}\mathrm{Proj}_{\mathrm{Ker}(M_2^1)}f_2^1(z, 0, \beta)$$

where $\mathrm{Proj}_S f$ is the projection of f on S, and

$$\mathrm{Ker}(M_2^1) = \mathrm{span}\left\{\begin{pmatrix} z_1\beta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\beta \end{pmatrix}\right\}.$$

After some computation, we obtain

$$\frac{1}{2}g_2^1(z, 0, \beta) = \begin{bmatrix} A_1 z_1 \beta \\ \overline{A_1} z_2 \beta \end{bmatrix}$$

with

$$A_1 = -cf'(0)e^{-i\omega_0\tau}u_2.$$

To compute the cubic terms, we first deduce that, after the change of variables that transformed the quadratic terms $f_2^1(z, y, \beta)$ into $g_2^1(z, y, \beta)$, the coefficients of the third order terms at $y = 0, \beta = 0$ are still given by $\frac{1}{6}f_3^1(z, 0, 0)$ (because $f''(0) = 0$, implying $f_2^1(z, y, 0) = 0$). This implies that [22]

$$\frac{1}{6}g_3^1(z, 0, \beta) = \frac{1}{6}\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}f_3^1(z, 0, \beta)$$

where

$$\mathrm{Ker}(M_3^1) = \mathrm{span}\left\{\begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1\beta^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\beta^2 \end{pmatrix}\right\}.$$

However, the terms $O(|z|\beta^2)$ are irrelevant to the determination of the generic Hopf bifurcation. Hence, we write

$$\frac{1}{6}g_3^1(z, 0, \beta) = \frac{1}{6}\mathrm{Proj}_S f_3^1\,(z, 0, 0) + O(|z|\beta^2)$$

for

$$S := \mathrm{span}\left\{\begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}\right\}.$$

Some computations yield

$$\frac{1}{6}g_3^1(z, 0, \beta) = \begin{bmatrix} A_2 z_1^2 z_2 \\ \overline{A_2} z_1 z_2^2 \end{bmatrix} + O(|z|\beta^2)$$

with

$$A_2 = \frac{cf'''(0)}{2}(1 - h_0 e^{-i\omega_0\tau})u_2.$$

Thus, we obtain the normal form (14) with the coefficients A_1, A_2 given explicitly in terms of the original equation (4), without the need to compute the center manifold beforehand. The normal form (14) can be written in real coordinates (x, y), through the change of variables $z_1 = x - iy, z_2 = x + iy$. In polar coordinates $(r, \theta), x = r\cos\theta, y = r\sin\theta$, this normal form becomes

$$\begin{cases} \dot{r} = K_1\beta r + K_2 r^3 + O(\beta^2 r + |(r, \beta)|^4) \\ \dot{\theta} = -\omega_0 + O(|(r, \beta)|) \end{cases} \tag{15}$$

where $K_1 := \mathrm{Re}\,A_1, K_2 := \mathrm{Re}\,A_2$.

We have the following theorem.

Theorem 3: In formula (15), the sign of K_1K_2 determines the direction of the Hopf bifurcation: if $K_1K_2 < 0\,(> 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $h > h_0\,(< h_0)$; the sign of K_2 determines the stability of the bifurcating periodic orbits: the bifurcating periodic orbits is stable (unstable) if $K_2 < 0\,(> 0)$.

4 A numerical example

Consider an example in the form of system (1), with $a = 1$, $b = 1.1$, $c = 1$, $\tau = 1$, and $f(.) = \tanh(.)$. The theoretical analysis in Section 2 leads to

$$\omega_0 = 0.9017, \qquad h_0 = 1.1496$$

It then follows from the results in Section 3 that

$$K_1 = 0.2627, \qquad K_2 = -0.3408.$$

These calculations prove that the equilibrium (0, 0) is stable when $h < h_0$, as shown by Figure 1, where $h = 1.1$. When h passes through the critical value $h_0 = 1.1496$, the equilibrium losses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the equilibrium. Each individual periodic orbit is stable for $K_2 < 0$. Since $K_1K_2 < 0$, the bifurcating periodic solutions exist at least for values of h slightly larger than the critical value. Choosing $h = 1.4$, as predicted by the theory, Figure 2 shows that there is an orbitally stable limit cycle.

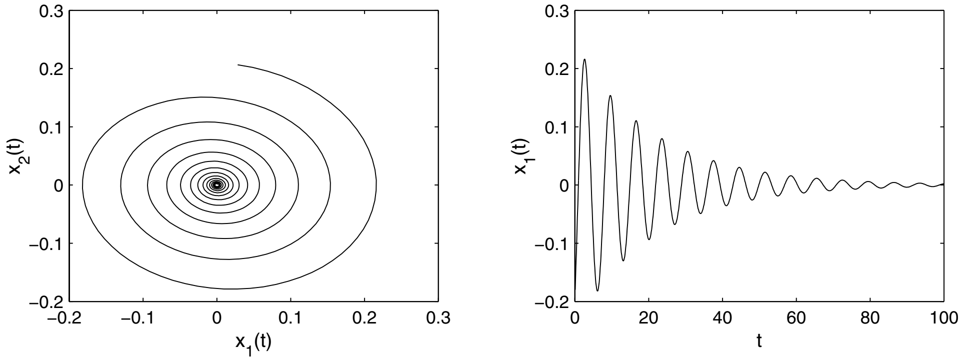

Fig. 1. Phase plot and waveform plot for system (1) with $h = 1.1$.

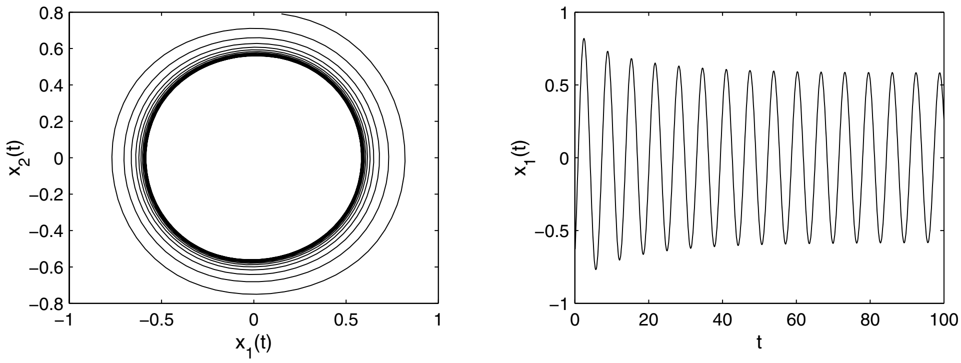

Fig. 2. Phase plot and waveform plot for system (1) with $h = 1.4$.

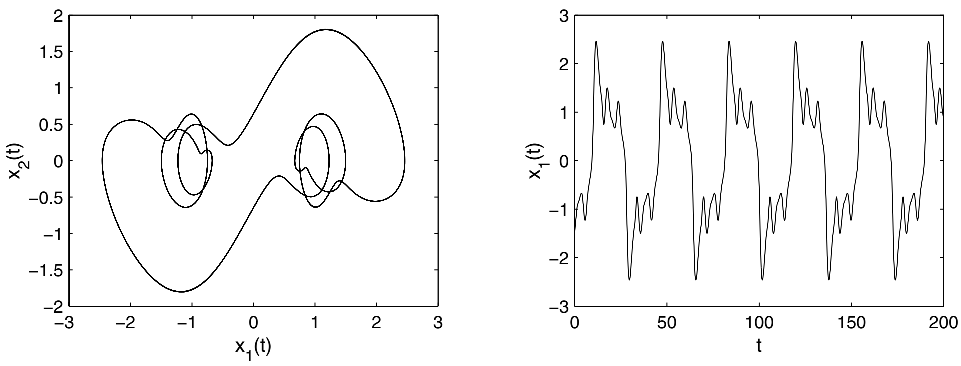

Fig. 3. Phase plot and waveform plot when $h = 0.7$.

5 Chaotic behavior

In this section, we study the dynamical behavior of system (1) with the activation function $f(x) = xe^{-x^2/2}$ and $a = 0.8, b = 1, c = 5, \tau = 5$, and let h be a variable parameter. It is noted that several other parameters have also been examined and shown to exhibit similar dynamical phenomena. Due to limitation of space, those results are not presented here.

When $h < 0.65$, the system is stable. When increasing h to $h = 0.7$, the system produces a periodic orbit. When $h = 0.7$, the phase plot and the waveform plot of $x_1(t)$ is shown in Figure 3. When the value of h passes 0.9, the system becomes chaotic. In Figure 4, we show the phase plot when $h = 1.0$, and in Figure 5 we show the waveform of $x_1(t)$ and the power spectrum plots

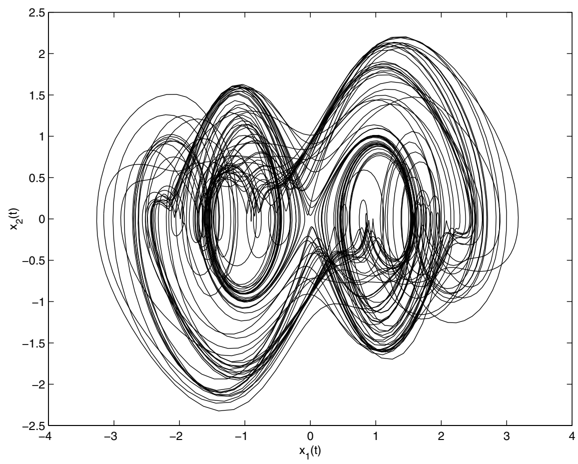

Fig. 4. Phase plot when $h = 1.0$.

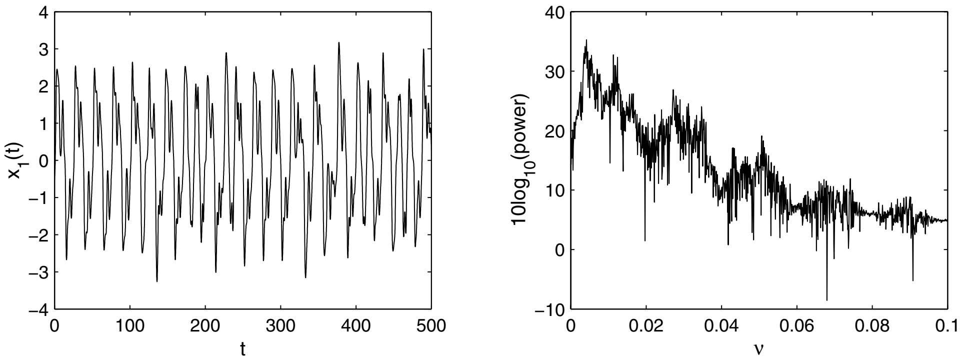

Fig. 5. Waveform and power spectrum plots when $h = 1.0$.

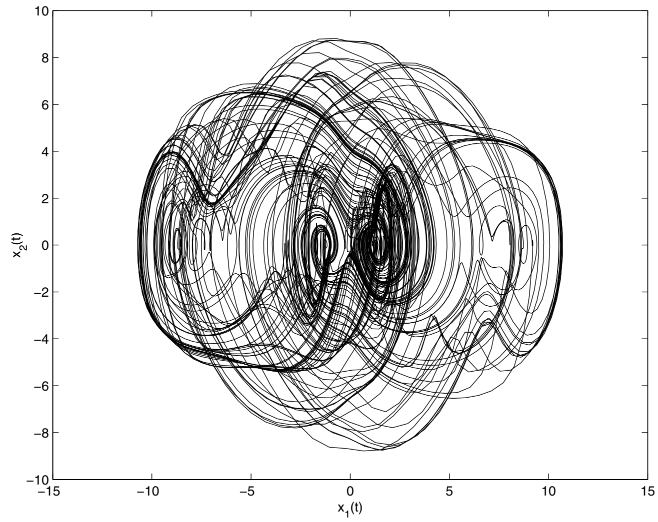

Fig. 6. Phase plot when $h = 5.0$.

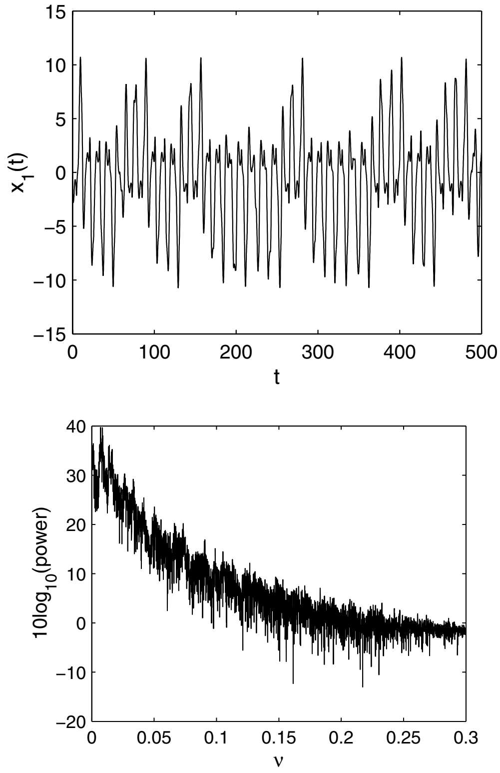

Fig. 7. Waveform and power spectrum plots when $h = 5.0$.

when $h = 1.0$. When $h = 5.0$, the phase plot and the waveform of $x_1(t)$, and the power spectrum plots, are shown in Figures 6 and 7, respectively. From these figures, we can see that the system is also chaotic, but it is different from the case of $h = 1.0$.

6 Conclusions

A single delayed neuron model with inertial term has been investigated in this paper. Hopf bifurcation is studied by using the normal form theory of retarded FDEs, in which the coefficients of the normal form are obtained in terms of the original delayed equation directly, without the need to compute the center manifold beforehand, which simplifies the computational procedure. With a nonmonotonic activation function, chaotic behavior has also been observed in this system.

This work was supported by the National Natural Science Foundation of China under Grant 60271019 and the Hong Kong Research Grants Council under the CERG grant CityU 1004/02E.

References

1. K.E. Kurten, J.W. Clark, Phys. Lett. A **144**, 413 (1986)
2. T.B. Kepler, S. Datt, R.B. Meyer, L.F. Abbott, Physica D **46**, 449 (1990)
3. P.K. Das II, W.C. Schieve, Z.J. Zeng, Phys. Lett. A **161**, 60 (1991)
4. F. Zou, J.A. Nossek, IEEE Trans. CAS **40**, 166 (1993)
5. C. Li, J. Yu, X. Liao, Phys. Lett. A **285**, 368 (2001)
6. V.E. Bondarenko, Phys. Lett. A **236**, 513 (1997)

7. T. Ueta, G. Chen, in *Controlling Chaos and Bifurcations in Engineering Systems,* edited by G. Chen (CRC Press, Boca Raton, 2000), pp. 581–601
8. L. Chen, K. Aihara, Neural Networks **8**, 915 (1995)
9. L. Chen, K. Aihara, IEEE Trans. CAS-I **47**, 1455 (2001)
10. C. Li, X. Liao, J. Yu, Neurocomputing **55**, 731 (2003)
11. K.L. Babcock, R.M. Westervelt, Physica D **28**, 305 (1987)
12. D.W. Wheeler, W.C. Schieve, Physica D **105**, 267 (1997)
13. X. Liao, K.-W. Wong, C.-S. Leung, Z. Wu, Chaos, Solitons and Fractals **12**, 1535 (2001)
14. X. Liao, Z. Wu, J. Yu, IEEE Trans. SMC-A **29**, 692 (1999)
15. X. Liao, K.-W. Wong, Z. Wu, Physica D **149**, 123 (2001)
16. M. Gilli, IEEE Trans. CAS **40**, 849 (1993)
17. J. Wei, S. Ruan, Physica D **130**, 255 (1999)
18. S. Ruan, J. Wei, Proc. Roy. Soc. Edingburgh Sect. A **129**, 1017 (1999)
19. L. Olien, J. Bélair, Physica D **102**, 349 (1997)
20. K. Gopalsamy, I. Leung, Physica D **89**, 395 (1996)
21. T. Faria, J. Diff. Equ. **168**, 129 (2000)
22. T. Faria, L.T. Magalhães, J. Diff. Equ. **122**, 181 (1995)
23. T. Faria, L.T. Magalhães, J. Diff. Equ. **122**, 224 (1995)
24. R. Bellman, K.L. Cooke, *Differential-Difference Equations* (Academic Press, New York, 1963)
25. J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations* (Springer-Verlag, New York, 1993)
26. K.L. Cooke, Z. Grossman, J. Math. Anal. Appl. **86**, 592 (1982)
27. S.A. Campbell, J. Bélair, T. Ohira, J. Milton, Chaos **5**, 640 (1995)
28. S.A. Campbell, J. Bélair, T. Ohira, J. Milton, J. Dynamics and Differential Equations **7**, 213 (1995)